

ON LIE'S THEOREM IN OPERATOR ALGEBRAS

BY

F.-H. VASILESCU

ABSTRACT. This work contains some algebraic results concerning infinite dimensional Lie algebras, as well as further statements within a topological background. Natural generalizations of the notion of radical, solvable and semi-simple Lie algebra are introduced. The last part deals with variants of a Lie's theorem in operator algebras.

1. Introduction. Let L be a Lie algebra, not necessarily finite dimensional, over a field K . Denote by $\{L^{(n)}\}$, $n \geq 0$, the derived series, i.e. $L^{(0)} = L$, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$, $n > 0$. We recall that L is said to be solvable when $L^{(n)} = 0$ for a certain n . It is well known [3] that every subalgebra and every homomorphic image of a solvable Lie algebra is solvable. Also the sum of two solvable ideals is a solvable ideal. In particular, when L is finite dimensional, there is a solvable ideal R_L of L which contains any other solvable ideal of L and this is the radical of L . In what follows we attempt to construct a radical for a not necessarily finite dimensional Lie algebra as an ideal R_L of L which contains any solvable ideal of L such that the quotient L/R_L has 0 as radical. Moreover, R_L is the smallest ideal in L with the described properties. We mention that there are some other constructions of a radical [6] but our point of view seems quite different. As we have a radical, we may define the notion of semisimple Lie algebra. It turns out that our definition is equivalent with the definition given in [4] for a particular infinite dimensional case.

Now, let L be a finite dimensional Lie algebra and K an algebraically closed field of characteristic zero. If L is solvable then there is a chain of ideals in L ,

$$0 = I_0 \subset I_1 \subset \dots \subset I_n \subset L = L,$$

such that $\dim I_k = k$ ($k = 1, \dots, n$; $n = \dim L$). This result is known as the Lie's theorem for abstract Lie algebras [3]. A more sophisticated but equivalent formulation of this result is the following: Every irreducible representation of a solvable Lie algebra on a finite dimensional vector space is necessarily one dimensional [5]. We investigate some infinite dimensional versions of this latter result, valid for operator algebras, as done in [7]. M. Şabac has given in [7] a

Received by the editors December 2, 1971.

AMS (MOS) subject classifications (1970). Primary 17B65, 47D15.

Key words and phrases. Radical, solvable Lie algebras, operator algebras.

Copyright © 1973, American Mathematical Society

definition for infinite dimensional solvable Lie algebras and has shown that for some special Lie algebras of linear operators, solvable in his sense, when irreducible the dimension of the space is one. We give here a more natural definition and get the definition in [7] as a corollary. At the same time that is a version of Lie's theorem for abstract Lie algebras. We also generalize the result in [7] by using similar techniques.

The author is grateful to S. Halperin for some helpful discussions.

2. The radical of a Lie algebra. Let L be a Lie algebra over a field K .

2.1. Definition. An ideal P of L is called primitive if for any ideal I of L with the property $I^{(1)} \subset P$ it follows that $I \subset P$. It is obvious that L itself is a primitive ideal. If P is primitive and I is an ideal of L such that $I^{(n)} \subset P$ for a certain $n \geq 0$ then $I \subset P$; therefore if P is primitive in L then P contains any solvable ideal of L . The intersection of any family of primitive ideals is a primitive ideal.

2.2. Lemma. *An ideal P of L is primitive if and only if the quotient L/P has no nontrivial abelian ideal.*

Proof. If P is primitive and I_1 is abelian in L/P then $\phi^{-1}(I_1) \subset P$, where ϕ is the canonical mapping of L onto L/P , hence $I_1 = 0$. Conversely, if L/P has no nontrivial abelian ideal and $I^{(1)} \subset P$, then $\phi(I) = 0$ and $I \subset P$.

2.3. Definition. The intersection R_L of all primitive ideals of L is called the radical of L .

Let us remark that if L is finite dimensional then R_L is the usual radical of L . For, it is sufficient to note that the largest solvable ideal of L is primitive.

2.4. Lemma. *If the only abelian ideal of L is 0 then $R_L = 0$.*

Proof. Indeed, 0 is in this case a primitive ideal of L .

2.5. Proposition. *If P is a primitive ideal of L then the radical of the quotient L/P is 0.*

Moreover, R_L is the smallest ideal I of L with the property that the radical of L/I is 0.

Proof. The assertion follow easily from Lemmas 2.2 and 2.4.

In case when L is finite dimensional we may give a complete description of its primitive ideals.

2.6. Proposition. *Let L be a finite dimensional Lie algebra over a field of characteristic zero. An ideal P of L is primitive if and only if $P \supset R_L$.*

Proof. It is sufficient to show that if $P \supset R_L$ then P is primitive. We recall that there is a semisimple subalgebra S of L (Levi subalgebra) such that for every ideal M of L we have

$$M = (R_L \cap M) + (S \cap M),$$

where the sum is direct (for details see [1]). Denote by p the projection of L on S . As R_L is an ideal then p is a homomorphism. We have $P = R_L + S \cap P$ and suppose that $I^{(1)} \subset P$, where I is an ideal in L . Then $pI^{(1)} = (pI)^{(1)} \subset S$ and S is semisimple, hence $(pI)^{(1)} = pI$. But $pI^{(1)} \subset S \cap P$, whence $I \subset R_L + S \cap P = P$, and P is primitive.

2.7. Proposition. *Let L be a finite dimensional Lie algebra over a field of characteristic zero. If I is an ideal of L and P is primitive in L then $P \cap I$ is primitive in I .*

Proof. Indeed, $P \cap I \supset R_L \cap I$ and $R_L \cap I$ is the radical of I [1], hence $P \cap I$ is primitive in I .

2.8. Definition. We say that L is semisimple when the radical of L is null. We say that L is simple when L is not commutative and the only ideals of L are 0 and L . Obviously, when L is simple then L is semisimple.

Let $\{H_\alpha\}_{\alpha \in A}$ be a family of ideals in L . Then by $\sum_{\alpha \in A} H_\alpha$ we mean the ideal generated in L by the family $\{H_\alpha\}_{\alpha \in A}$.

2.9. Proposition. *If $L = \sum_{\alpha \in A} L_\alpha$ and L_α are simple ideals of L then L is semisimple.*

Moreover, if I is an ideal of L then there is a set of indices $B \subset A$ such that $I = \sum_{\beta \in B} L_\beta$.

Proof. The proof of this result is quite standard [1] and the second part involves obviously the first one. Let us sketch the proof of the second part. Suppose $L = \sum_{\alpha \in A} L_\alpha$, where we may assume $L_\alpha \cap L_\beta = 0$ ($\alpha \neq \beta$). If $0 \neq I \subset L$ is an ideal, then for any α we have either $L_\alpha \subset I$ or $L_\alpha \cap I = 0$. Let B be the set of indices $\beta \in A$ such that $L_\beta \subset I$. It is easy to see that for any $\alpha \in A$ the following equality holds:

$$\{x \in L; [x, L_\alpha] = 0\} = \sum_{\gamma \neq \alpha} L_\gamma.$$

Take an $x \in I$, $x = x_{\alpha_1} + \cdots + x_{\alpha_n}$, with $x_{\alpha_j} \in L_{\alpha_j}$ ($j = 1, \dots, n$). If $L_{\alpha_j} \cap I = 0$ then $[L_{\alpha_j}, I] = 0$ and $x_{\alpha_j} = 0$; therefore $I = \sum_{\beta \in B} L_\beta$.

When L is an L^* -algebra then L is semisimple in the sense of Definition 2.8 if and only if it is semisimple as defined in [4]. We recall that an L^* -algebra is a complex Hilbert space which is also a Lie algebra, with the property that for any $x \in L$ there is an $x^* \in L$ such that $([x, y], z) = (y, [x^*, z])$, for all $y, z \in L$ (where (x, y) is a scalar product of L). In such an algebra, the Lie product is continuous with respect to the topology of L [4].

For any set $H \subset L$ denote by \bar{H} its closure in the topology of L .

2.10. Proposition. *Let L be an L^* -algebra. Then L is semisimple if and only if $L = \overline{L^{(1)}}$.*

Proof. We use some results in [4]. First, every L^* -algebra is the direct sum of an abelian ideal and $\overline{L^{(1)}}$; therefore if $R_L = 0$ then $L = \overline{L^{(1)}}$. Conversely, if $L = \overline{L^{(1)}}$ then $L = \sum_{\alpha} L_{\alpha}$, where each L_{α} is a noncommutative closed ideal of L which does not contain any proper closed ideal, and the sum is direct in the sense of Hilbert spaces. Noting that an ideal I is commutative if and only if \overline{I} is commutative, we deduce easily that L has no commutative ideal. By Lemma 2.4 we have $R_L = 0$.

Any L^* -algebra with the property $L = \overline{L^{(1)}}$ is called in [4] semisimple. Proposition 2.10 shows that our notion is equivalent with the Schue's concept in the class of L^* -algebras.

3. Quasi-solvable algebras. We are looking for a class of Lie algebras corresponding to the class of solvable algebras in the finite dimensional case, but with some infinite dimensional characteristics.

3.1. Definition. An ideal I of L is quasi-solvable in L if $I = \sum_{\alpha} I_{\alpha}$, where I_{α} is a solvable ideal of L for every α . More general, the algebra L is quasi-solvable if L is quasi-solvable as ideal of L .

3.2. Proposition. *If L is a quasi-solvable algebra then one has $R_L = L$.*

3.3. Proposition. *Let L be a quasi-solvable algebra. Then any subalgebra and any homomorphic image of L is quasi-solvable.*

Propositions 3.1 and 3.2 have a very simple proof.

3.4. Proposition. *Let L be a quasi-solvable algebra over an algebraically closed field of characteristic zero and I a finite dimensional ideal of L . Then there is a chain $\{I_k\}_{k=0}^n$ of ideals of L with $\dim I_k = k$, where $n = \dim I$, such that*

$$0 = I_0 \subset I_1 \subset \dots \subset I_n = I.$$

Proof. Let $\phi(x)$ be the restriction of the mapping $D_x(y) = [x, y]$ on I . Then $x \rightarrow \phi(x)$ is a finite dimensional representation of L . If $L_1 = \phi(L)$ then, by Proposition 3.3, L_1 is quasi-solvable and finite dimensional, hence solvable. By Lie's theorem, there is an increasing chain $\{I_k\}_{k=0}^n$ such that I_k are linear subspaces of I , $\dim I_k = k$ and $D_x(I_k) \subset I_k$ ($k = 1, \dots, n$).

3.5. Definition. Let L be a Lie algebra over an algebraically closed field of characteristic zero. We say that L is σ -solvable if L is generated by a countable family of finite dimensional solvable ideals.

It is obvious that any σ -solvable algebra is quasi-solvable. The next result may be considered as a Lie's theorem for abstract Lie algebras.

3.6. Theorem. *Let L be a σ -solvable Lie algebra. Then there is a sequence $\{I_n\}_{n \geq 0}$ of finite dimensional solvable ideals of L with the properties:*

- (i) $\dim I_n = n$ ($n \geq 0$);
- (ii) $0 = I_0 \subset I_1 \subset \dots \subset I_n \subset \dots \subset L$;
- (iii) $\bigcup_{n \geq 0} I_n = L$.

Proof. We may suppose from the very beginning that $L = \bigcup_m H_m$ where each H_m is a finite dimensional solvable ideal of L , and the sequence $\{H_m\}_{m \geq 1}$ is increasing. We suppose also that $H_m \neq H_{m+1}$ for every $m > 0$. If $\dim H_1 = 1$ then we put $I_1 = H_1$; if not, then we apply Proposition 3.4 to H_1 and get a chain

$$0 = I_0 \subset I_1 \subset \dots \subset I_{n_1} = H_1,$$

where each I_j is an ideal of L , $\dim I_j = j$ and $n_1 = \dim H_1$. Take now $L_1 = L/H_1$ and consider the ideal $\phi_1(H_2)$ in L_1 , where ϕ_1 is the canonical mapping of L onto L_1 . If $\dim \phi_1(H_2) = 1$, we put $I_{n_1+1} = H_2$; when $\dim \phi_1(H_2) = n_2 > 1$, then we apply again Proposition 3.4 to $\phi_1(H_2)$ and get an increasing chain $\{I_{1,j}\}_{j=1}^{n_2}$ of ideals in L_1 such that $\dim I_{1,j} = j$ and define $I_{n_1+l} = \phi_1^{-1}(I_{1,l})$ ($0 < l \leq n_2$). This method can be continued and we obtain a sequence $\{I_n\}_{n \geq 0}$ with the desired properties.

The properties (i), (ii), and (iii) are used in [7] in order to define ∞ -solvable algebras. The above theorem shows that it is possible to have a more concise definition.

4. σ -solvable algebras of operators. In this section X will be a fixed complex Banach space and $B(X)$ the algebra of all linear continuous operators on X .

Let L be a σ -solvable Lie algebra and θ a representation of L in X , i.e. θ is a homomorphism of L into $B(X)$. It is easy to see that $\theta(L)$ is also a σ -solvable Lie algebra, therefore when such features are concerned we may deal only with σ -solvable subalgebras of $B(X)$.

The proof of the next result has some common ideas with one in [7].

4.1. Theorem. *Let L be a σ -solvable Lie algebra in $B(X)$. Then either L is commutative or L contains a non-null finite dimensional ideal Q such that the elements of Q are commuting quasi-nilpotent operators.*

Proof. Suppose that there is no non-null finite dimensional ideal of L whose elements are commuting quasi-nilpotent operators. As L is σ -solvable, according to Theorem 3.6 there is an increasing sequence of solvable ideals $\{I_n\}_{n \geq 0}$ such that $\dim I_n = n$ and $\bigcup_n I_n = L$. We prove by recurrence with respect to n that $[I_n, L] = 0$ for any $n \geq 1$. When $n = 1$, as $\dim I_1 = 1$ then there is an $a_1 \in I_1$, $a_1 \neq 0$, such that $[x, a_1] = \lambda_1 a_1$, for any $x \in L$. Suppose $\lambda_1 \neq 0$ for a certain $x \in L$. Then, according to the Kleinecke-Sirokov theorem [2], the operator $a_1 = \lambda_1^{-1}[x, a_1]$ is quasi-nilpotent, hence I_1 is a one-dimensional ideal of L having a

quasi-nilpotent generator and, by our assumption, this is impossible. Suppose now $[I_k, L] = 0$ and let us show that $[I_{k+1}, L] = 0$. For, let us choose a basis a_1, \dots, a_k, a_{k+1} in I_{k+1} such that $a_1, \dots, a_k \in I_k$. Then for any $x \in L$ we have

$$[a_{k+1}, x] = \eta_1 a_1 + \dots + \eta_k a_k + \eta_{k+1} a_{k+1}.$$

As $[I_k, L] = 0$, we have $[a_{k+1}, [a_{k+1}, x]] = 0$, hence $q_x = [a_{k+1}, x]$ is quasi-nilpotent, according again to the Kleinecke-Sirokov theorem. If $q_x = 0$ for any $x \in L$ then $[I_{k+1}, L] = 0$ and this is just what we need. If $q_x \neq 0$ for at least one $x \in L$, denote by Q the non-null set of quasi-nilpotent elements of I_{k+1} . We have obviously $[I_{k+1}, I_{k+1}] = 0$, hence the elements in Q commute. If $q_1, \dots, q_m \in Q$ then $\alpha_1 q_1 + \dots + \alpha_m q_m$ is again quasi-nilpotent, hence Q is a linear subspace. We have also $[L, Q] \subset [L, I_{k+1}] \subset Q$, hence Q is an ideal of L and this is impossible. Consequently, L must contain only commuting elements.

4.2. Corollary. *Let L be a Lie algebra in $B(X)$ such that R_L is σ -solvable. If the associative algebra with unit generated by L in $B(X)$ is semisimple then R_L is equal to the center of L .*

Proof. Denote by C_L the center of L . Since C_L is commutative we have $C_L \subset R_L$. On the other hand, as the associative algebra generated by L in $B(X)$ is semisimple, R_L may not contain any non-null quasi-nilpotent operator. By Theorem 4.1, R_L is commutative. Now, if $x \in L$ and $y \in R_L$ then $[x, y] \in R_L$ and $[[x, y], y] = 0$. By the Kleinecke-Sirokov theorem it follows that $[x, y]$ is quasi-nilpotent, therefore $[x, y] = 0$. Consequently $R_L = C_L$.

4.3. Corollary. *If L is a Lie algebra in $B(X)$ such that the associative algebra with unit generated by L in $B(X)$ is semisimple, then L is σ -solvable if and only if L is commutative and has a countable algebraic basis.*

Proof. If L is commutative and has a countable basis then L is σ -solvable. Conversely, if L is σ -solvable then L has a countable basis and L may not contain any quasi-nilpotent operator, hence because of Theorem 4.1, it is commutative.

From now on we investigate some infinite dimensional variants of the following statement: Any finite dimensional irreducible representation of a solvable Lie algebra is one dimensional.

Let L be a complex Lie algebra and θ a representation of L in the Banach space X . We say that θ is irreducible [7] if there is no Banach space Y in X , $0 \neq Y \neq X$, such that $\theta(x)Y \subset Y$ for any $x \in L$. A question arises naturally: Is every irreducible representation of a σ -solvable Lie algebra one dimensional? The answer seems to be related more or less to the classical problem of invariant subspaces. A partial answer was given in [7]. We are going to generalize that result and to point out some other remarks.

4.4. Definition. A representation θ of a σ -solvable Lie algebra L in the Banach space X is called a d -representation if

- (1) $\theta(x)$ is a decomposable operator [2] on X for any $x \in L$.
- (2) If $\theta(x) - \lambda$ is quasi-nilpotent then $\theta(x) - \lambda$ is actually a nilpotent operator, where λ is a scalar.

4.5. Theorem. Let L be a σ -solvable complex Lie algebra and θ a d -representation of L in the Banach space X . If θ is irreducible then $\dim X = 1$.

Proof. We may suppose $L \subset B(X)$. Since L is σ -solvable, because of Theorem 4.1 we have to study two cases:

(a) Suppose L is commutative and denote by $\sigma(x)$ the spectrum of x as operator, where $x \in L$. If there is an $x_0 \in L$ such that $\sigma(x_0)$ contains at least two points, then there is a maximal spectral space of x_0 [2] which is different from zero and X , say Y . Since L is commutative we have $xY \subset Y$ for any $x \in L$ [2]. Consequently, Y is a proper Banach subspace of X invariant to L , and this is impossible. Therefore, for each $x \in L$ there is a complex number λ_x such that $\sigma(x) = \{\lambda_x\}$, i.e. $x - \lambda_x$ is quasi-nilpotent. By our assumption, $x - \lambda_x$ is actually nilpotent. If $x \neq \lambda_x$ for a certain x then $Y = \{\xi \in X; (x - \lambda_x)\xi = 0\}$ is a proper Banach subspace of X , invariant to L . This is again impossible, hence $x = \lambda_x$ for any $x \in L$. But in this case any closed subspace of X is invariant to L , thus we must have $\dim X = 1$.

(b) Suppose that there is a finite dimensional ideal Q of L such that Q contains only commuting quasi-nilpotent operators, hence nilpotent. Let us take a basis $\{q_1, \dots, q_n\}$ in Q and define the space

$$Y = \bigcap_{k=1}^n \ker q_k.$$

Since $q_k \neq 0$ for any k we have $Y \neq X$. On the other hand, since q_k ($k = 1, \dots, n$) are commuting nilpotents, $Y \neq 0$. We have also that Q is an ideal of L , hence for any $x \in L$ and $q \in Q$ there are some complex numbers α_j^l such that

$$q_l x = x q_l + \sum_{j=1}^n \alpha_j^l q_j \quad (l = 1, \dots, n),$$

therefore if $\xi \in Y$ then $x\xi \in Y$, hence Y is invariant to L and this is not possible. Anyway, we must have $\dim X = 1$ and the proof is complete.

4.6. Corollary [7]. Let L be a σ -solvable complex Lie algebra and θ a representation of L in the Banach space X , such that $\theta(x)$ is a scalar generalized operator [2] for any $x \in L$. If θ is irreducible then $\dim X = 1$.

Proof. It is sufficient to show that θ is a d -representation of L in the Banach space X . Indeed, every $\theta(x)$ is scalar generalized hence decomposable [2] and any quasi-nilpotent scalar generalized operator is actually nilpotent [2].

4.7. Proposition. *Let L be a σ -solvable complex Lie algebra and θ a representation of L in X with the properties:*

- (1) *$\theta(x)$ is decomposable for any $x \in X$.*
- (2) *The associative algebra with unit generated by $\theta(L)$ in $B(X)$ is semisimple.*

If θ is irreducible then $\dim X = 1$.

Proof. Assume $L \subset B(X)$. Since the associative algebra generated by L in $B(X)$ is semisimple, according to Corollary 4.3, we have L commutative. As in the proof of Theorem 4.5, if $\sigma(x_0)$ has at least two points, for a certain $x_0 \in L$, then there is a proper closed subspace of X , invariant to L , and this is not possible. If $\sigma(x) = \{\lambda_x\}$ for any $x \in L$ then $x - \lambda_x$ is quasi-nilpotent and, by our assumption, we should have $x = \lambda_x$. But in this case any closed subspace of X is invariant to L , unless $\dim X = 1$.

Acknowledgement. The author is grateful for the support of National Research Council Grant of Dr. A. J. Coleman, Head, Department of Mathematics, Queen's University. He would like also to thank the referee for many valuable suggestions which improved the algebraic part of this work.

REFERENCES

1. N. Bourbaki, *Éléments de mathématique*. XXVI. *Groupes et algèbres de Lie*. Chap. 1: *Algèbres de Lie*, Actualités Sci. Indust., no. 1285, Hermann, Paris, 1960. MR 24 #A2641.
2. I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.
3. N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. 10, Interscience, New York, 1962. MR 26 #166.
4. J. R. Schue, *Hilbert space methods in the theory of Lie algebras*, Trans. Amer. Math. Soc. 95 (1960), 69–80. MR 22 #8352.
5. Séminaire "Sophus Lie" de l'Ecole Normale Supérieure 1954/55. *Théorie des algèbres de Lie. Topologie des groupes de Lie*, Secrétariat mathématique, Paris, 1955. MR 17, 384.
6. I. Stewart, *Lie algebras*, Lecture Notes in Math., vol. 127, Springer-Verlag, Berlin and New York, 1970. MR 41 #8483.
7. M. Şabac, *Une généralisation du théorème de Lie*, Bull. Sci. Math. 95 (1971), 53–57.
8. F.-H. Vasilescu, *Radical d'une algèbre de Lie*, C. R. Acad. Sci. Paris 274 (1972), 536–538.

INSTITUTE OF MATHEMATICS, ACADEMY OF THE ROMANIAN SOCIALIST REPUBLIC,
BUCHAREST, ROMANIA (Current address)

DEPARTMENT OF MATHEMATICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA